Propositional Logic
Properties

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Outline

1 Validity and satisfiability

2 Equivalence

3 Entailment

4 Exercises

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2 Equivalence

3 Entailment

4 Exercises

Validity and satisfiability (1)

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- **valid** if all interpretations are models of ϕ

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1. Every tautology is satisfiable
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3. The converse of 1 and 2 does not hold in general

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Observations

1. Every tautology is satisfiable
2. Every contradiction is invalid
3. The converse of 1 and 2 does not hold in general
4. Tautologies and contradictions form disjoint sets

Validity and satisfiability (2)

Example

A	B	C	$\neg A$	$\neg A \vee B$	$\neg A \wedge C$	$\neg A \vee B \rightarrow \neg A \wedge C$
0	0	0	1	1	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	0	0	1
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- $\neg A \vee B \rightarrow \neg A \wedge C$ is satisfiable

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- $\neg A \vee B \rightarrow \neg A \wedge C$ is invalid
- $\neg A \vee B \rightarrow \neg A \wedge C$ is satisfiable
- $\neg A \vee B \rightarrow \neg A \wedge C$ is neither a tautology nor a contradiction

Validity and satisfiability (3)

Examples

Identify tautologies and contradictions among the following formulas:

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- $\phi \vee \neg\phi$ tautology
- $\phi \vee \top$

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- A wff ϕ is a tautology if and only if $\phi \equiv \top$

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Equivalence of ϕ and ψ is denoted as $\phi \equiv \psi$

Observations

- A wff ϕ is a tautology if and only if $\phi \equiv \top$
- A wff ϕ is a contradiction if and only if $\phi \equiv \bot$

Equivalence (2)

Example

- $\phi \vee \psi \equiv \psi \vee \phi$
- $\phi \wedge \psi \equiv \psi \wedge \phi$
- $\phi \leftrightarrow \psi \equiv \psi \leftrightarrow \phi$

Equivalence (2)

Example

- $\phi \vee \psi \equiv \psi \vee \phi$ (commutativity)
- $\phi \wedge \psi \equiv \psi \wedge \phi$ (commutativity)
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- $\phi \vee \top \equiv \top$
- $\phi \wedge \bot \equiv \bot$

Equivalence (3)

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- $\phi \vee \bot \equiv \phi$
- $\phi \wedge \top \equiv \phi$

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Example

- $\phi \vee \bot \equiv \phi$ (neutrality)
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- $\phi \vee \bot \equiv \phi$ (neutrality)
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- $\neg\neg\phi \equiv \phi$
- $\phi \rightarrow \psi \equiv \neg\phi \vee \psi$

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- $\phi \wedge \top \equiv \phi$ (neutrality)
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- $\neg\neg\phi \equiv \phi$
- $\phi \rightarrow \psi \equiv \neg\phi \vee \psi$
- $\phi \rightarrow \psi \equiv \neg\psi \rightarrow \neg\phi$ (contraposition)

Equivalence (4)

Example

- $\neg(\phi \vee \psi) \equiv \neg\psi \wedge \neg\phi$
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- $\phi \wedge (\phi \vee \psi) \equiv \phi$ (absorption)
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Definition (Model of a set of wffs)

Given an interpretation I and a set of wffs Γ,
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Warning!

- A set Γ of wffs could be seen as the conjunction of its formulas
- But this conjunction is not a wff in general!
- Actually, it is a wff only if Γ has finite cardinality

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For $I = \{T_1, T_2, \ldots\}$, consider $\Gamma = \{T_1, T_2, \ldots, \neg F_1, \neg F_2, \ldots\}$, where $F_1, F_2, \ldots$ are the false variables according to I

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- If $\Gamma \models \phi$ then we say that Γ entails ϕ
- We also say that ϕ is a logical consequence of Γ

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- If $\Gamma \models \phi$ then we say that Γ entails ϕ
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$\models \phi$ if and only if ϕ is a tautology

Entailment

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- $\{\phi, \psi\} \models \phi \rightarrow \psi$
- $\{\phi\} \models \phi \vee \psi$

Entailment

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- $\{\phi \wedge \psi\} \models \phi \rightarrow \psi$
- $\{\neg\phi\} \models \phi \rightarrow \psi$
- $\{\phi, \psi\} \models \phi \rightarrow \psi$
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Entailment

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If $\Gamma \models \phi$ *then* $\Gamma, \psi \models \phi$

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$\Gamma, \phi \models \psi$ if and only if $\Gamma \models \phi \rightarrow \psi$

Contraposition Theorem

$\Gamma \models \phi \rightarrow \psi$ if and only if $\Gamma, \neg\psi \models \neg\phi$

Outline

1 Validity and satisfiability

2 Equivalence

3 Entailment

4 Exercises

Exercises (1)

1. Practice with well-formed formulas at `http://www.inf.unibz.it/~nutt/Proplog/wellFormed.html`
2. Decide whether formula
$$x \vee \neg y \wedge \neg z$$
is satisfiable
3. Decide whether formula
$$\neg x \vee z \leftrightarrow y \wedge \neg z$$
is a tautology
4. Decide whether formula
$$x \leftrightarrow \neg y \rightarrow \neg z \wedge \neg x \vee y$$
is a contradiction

END OF THE LECTURE