Propositional Logic
Propositional Tableau and Sequent Calculus

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Outline

1 Propositional Tableau

2 Sequent Calculus

3 Exercises

Propositional Tableau (1)

- Yet another decision procedure for unsatisfiability
- The idea is to break down logical connectives into their constituent parts
- We consider the tableau for formulas in Negation Normal Form (NNF)

Negation Normal Form

1. Eliminate $\top$, $\bot$, $\rightarrow$, $\leftrightarrow$
2. Apply double negation equivalences:
   - $\neg\neg\phi \equiv \phi$
3. Apply De Morgan equivalences:
   - $\neg(\phi \vee \psi) \equiv \neg\psi \wedge \neg\phi$
   - $\neg(\phi \wedge \psi) \equiv \neg\psi \vee \neg\phi$

Propositional Tableau (2)

Let Γ be a set of wffs in NNF.

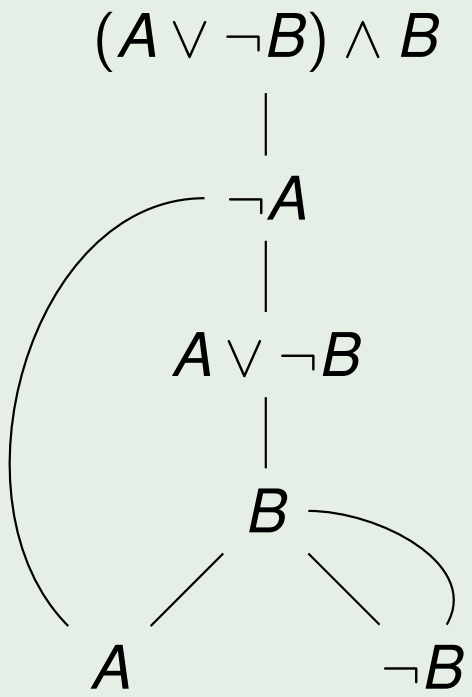
- Start with a chain made of the formulas in Γ
- If a branch contains a formula $\phi \wedge \psi$ then add to its leaf the chain ϕ — ψ
- If a branch contains a formula $\phi \vee \psi$ then add to its leaf two children ϕ and ψ
- If a branch contains a formula and its negation then close the branch
- If all branches are closed then Γ is unsatisfiable

Notation

$$\frac{\phi \wedge \psi}{\begin{array}{c}\phi \\ \psi\end{array}}\,(\wedge) \qquad \frac{\phi \vee \psi}{\phi \mid \psi}\,(\vee)$$

Propositional Tableau (3)

Example: $\Gamma := \{(A \vee \neg B) \wedge B, \neg A\}$



Γ is unsatisfiable!

We can simplify the notation used for a closure by putting a marker on the leaf of the closed path!

On the blackboard

$A \wedge C, \neg A \vee B$

Theorem

Propositional tableau is sound and complete, i.e.,
if the tableau for Γ is closed then $\Gamma \models \bot$, and
if $\Gamma \models \bot$ then the tableau for Γ is closed.

Sequent Calculus

Idea

Define inference rules for sequents

$$\Gamma \vdash \Delta$$

where Γ and Δ are sequences of formulas

Intuition Read $\Gamma \vdash \Delta$ like $(\bigwedge \Gamma) \to (\bigvee \Delta)$
Goal 1 $\Gamma \vdash \Delta$ holds if $\Gamma \models \Delta$ (completeness)
Goal 2 $\Gamma \vdash \Delta$ implies $\Gamma \models \Delta$ (soundness)

Notation

$$\frac{S_1}{S} \quad \text{or} \quad \frac{S_1 \quad S_2}{S}$$

From sequents S_1 (and S_2) conclude sequent S.

System considered here: **LK**, defined by Gerhard Gentzen

LK — Logistischer Klassischer Kalkül (1)

Axioms

$$\frac{}{A \vdash A}\,(ax)$$

What are A and B?
What $\Gamma, \Gamma', \Delta, \Delta'$?

Structural rules

permutation

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,(p.l) \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,(p.r)$$

contraction

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(c.l) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,(c.r)$$

weakening

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,(w.l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,(w.r)$$

cut

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(cut)$$

Logical rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,(\wedge l.1) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,(\wedge r)$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,(\wedge l.2)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,(\vee l) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\,(\vee r.1)$$

$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\,(\vee r.2)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}\,(\neg l) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,(\neg r)$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to l) \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\,(\to r)$$

Sequent Calculus

- Use these inference rules consecutively

Example

$$\dfrac{\dfrac{}{A \vdash A}\,(ax)}{\vdash A, \neg A}\,(\neg r)$$

- If on top there are only axioms then it is a derivation of the bottom sequent

Theorem

Sequent Calculus is sound and complete, i.e.,
if we can derive $\Gamma \vdash \Delta$ then $\Gamma \models \Delta$, and
if $\Gamma \models \Delta$ then there is a derivation for $\Gamma \vdash \Delta$.

LK — Summary

$$\frac{}{A \vdash A}\,(ax) \qquad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,(p.l) \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,(p.r) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,(w.l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,(w.r)$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(c.l) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,(c.r) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,(cut)$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,(\wedge l.1) \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\,(\wedge r)$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\,(\wedge l.2)$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,(\vee l) \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B}\,(\vee r.1)$$

$$\frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}\,(\vee r.2)$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}\,(\neg l) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,(\neg r)$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\,(\to l) \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\,(\to r)$$

LK — Example

Example

1. $(A \lor \neg B) \land B, \neg A \models \bot$
2. $A \land C, \neg A \lor B$

(On the blackboard.)

Exercises (1)

(From *Logic for Computer Science: Foundations of Automatic Theorem Proving*)

Give proof trees for the following tautologies:

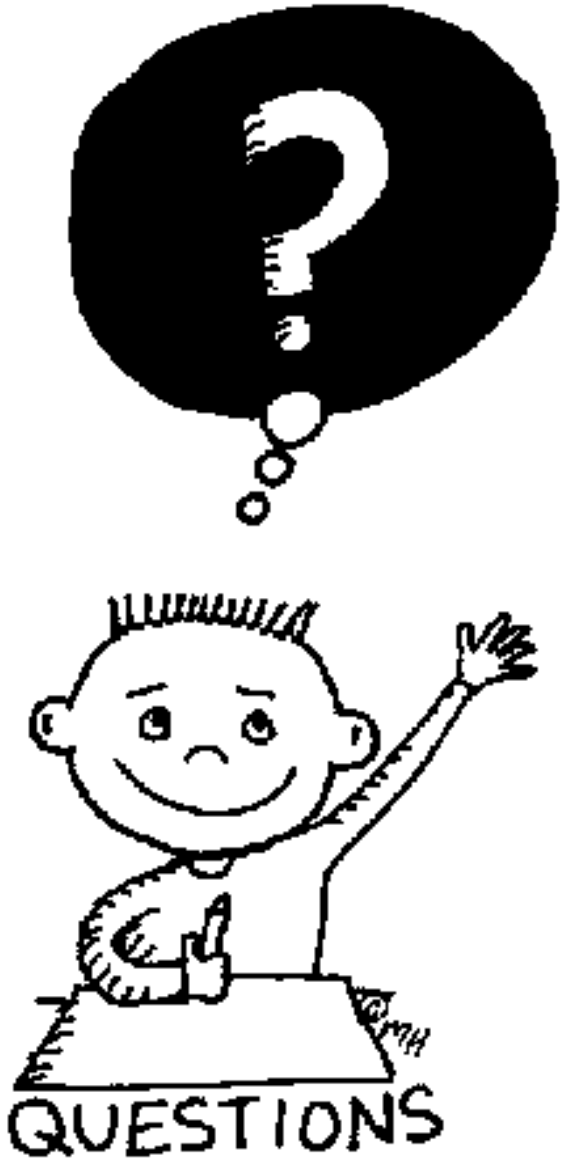
1. $A \to (B \to A)$
2. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
3. $A \to (B \to A \wedge B)$
4. $A \to A \vee B$
5. $B \to A \vee B$
6. $(A \to B) \to ((A \to \neg B) \to \neg A)$
7. $A \wedge B \to A$
8. $A \wedge B \to B$
9. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$
10. $\neg\neg A \to A$

Exercises (2)

(From *Logic for Computer Science: Foundations of Automatic Theorem Proving*)

Give proof trees for the following equivalences:

1. $(A \vee B) \vee C \equiv A \vee (B \vee C)$ (associativity)
2. $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ (associativity)
3. $A \vee B \equiv B \vee A$ (commutativity)
4. $A \wedge B \equiv B \wedge A$ (commutativity)
5. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ (distributivity)
6. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ (distributivity)
7. $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (De Morgan)
8. $\neg(A \wedge B) \equiv \neg A \vee \neg B$ (De Morgan)
9. $A \vee A \equiv A$ (idempotency)
10. $A \wedge A \equiv A$ (idempotency)
11. $\neg\neg A \equiv A$ (double negation)
12. $(A \vee B) \wedge (\neg B \vee C) \equiv (A \vee B) \wedge (\neg B \vee C) \wedge (A \vee C)$ (resolution)



END OF THE LECTURE